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ABSENCE OF CORPUS CALLOSUM, A HEREDITARY BRAIN ANOMALY OF THE HOUSE MOUSE. PRELIMINARY REPORT

By Lester S. King and Clyde E. Keeler

From the Department of Neuropathology and the Howe Laboratory, Harvard Medical School

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In 1923 there was discovered in mice a hereditary anomaly of the eye known as Rodless Retina.[1] The retinæ of rodless animals were found to lack visual end organs (rods) and to have external nuclei greatly reduced in number and abnormal in differentiation. The ganglionic cells and the optic nerve fibres were histologically normal.[2] The retina does not serve visual function, for there was no action current in response to light stimulus[3] and training tests showed total blindness.[4,5] Rodless Retina is inherited as a simple, recessive Mendelizing unit-character, devoid of any lethal or deleterious action.[6] Through the facilities provided by the Bussey Institution it has been possible to develop and perpetuate a strain of these rodless mice for further investigation.

Anatomical studies were made of the brains of these rodless mice, with special reference to the visual areas. These will be reported by one of us in the near future. In this investigation it was found that several of the brains showed a complete absence of the corpus callosum, which is the commissure of the neo-cortex.

According to Kappers[7] this commissure first appears in the monotremes. But in these animals, as well as in most of the marsupials,[8] the commissural fibres do not have an independent position of their own. Instead, they pass in the anterior commissure, the connection between the paleo-cortex of each hemisphere. A separate dorsal corpus callosum first appears in the edentates,[9] and in all higher orders develops in direct ratio to the degree of cerebral convolution. Its greatest development is in man. Since the literature on the corpus callosum is extensive, no attempt to review it will be made in this preliminary note. Suffice it to say that numerous cases of partial or total absence of the corpus callosum in man have been reported together with associated abnormalities. Tumbelaka[10] has reported the total absence of corpus callosum in a Cebus monkey.

Anatomy.—The anatomical features are shown in the accompanying figures. There is no other gross abnormality associated with this condition, and other tracts approximate the normal.

Figure 1 shows a cross-section through the cerebral hemispheres of the normal mouse, at the level of the foramen of Munro (*FM*). All three commissures of the forebrain are shown. The anterior commissure (*AC*) is cut twice, once in its central portion, surmounted by the descending columns of the fornix (*F*), and also in the temporal limb. The hippocampal commissure (*H*) with the corpus callosum (*CC*) above it, display the structure typical of rodents. In addition, the section shows the centrum ovale (*CO*), internal capsule (*IC*), terminal striæ (*TS*), lateral olfactory tract (*OT*) and optic chiasm (*OC*).

Figure 2 passes through the same level in the abnormal brain. The corpus callosum is absent. The longitudinal fissure (*LF*) extends down

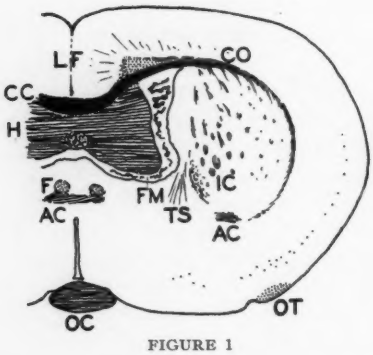

FIGURE 1

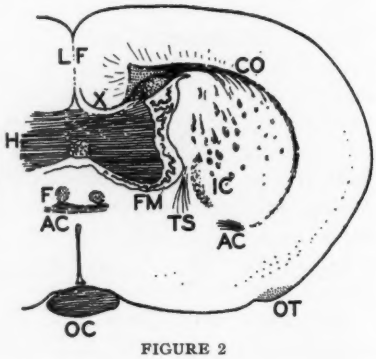

FIGURE 2

to the hippocampal commissure, with no dorsal connection of the hemispheres. In sections caudad to the one depicted, the longitudinal fissure extends without interruption to the roof of the third ventricle. At *X* is shown an aberrant bundle of fibres which is inconstant in the abnormal brains examined. It has no homologue in the normal brain, and may represent the abortive analage of the corpus callosum.

Reactions.—We have not been able, by casual observation, to distinguish between normal mice and those lacking the corpus callosum.

Heredity.—It was important to determine whether the absence of corpus callosum represented merely a somatic variation or whether it had a hereditary basis. The examination of a number of brains from the rodless stock upon which the brain study had been made revealed additional animals lacking the corpus callosum.

We desired to learn whether the character is associated with rodless

retina, and, if not, whether it is inherited at all. In event of heredity and dissociation, we wished to learn whether it segregated definitely from the normal, and whether it could be selected to provide material for additional studies.

For these reasons there were examined brain sections from members of a number of families of mice originating from an outcross of the rodless strain. Some of these were examined histologically for the rodless variation, and some were recorded for the presence or absence of several other hereditary characters.

This survey showed that:

1. Absence of corpus callosum segregates sharply. Twenty-four cases were observed, none of which were intermediate between the condition as described above and the normal structure.

2. The character is present in several families of like origin. Samples of nine cages segregated both normal and abnormal animals, whereas samples from four cages contained only animals lacking the corpus callosum. Samples from two other cages of the same origin were completely normal, suggesting that they may be breeding true for the presence of corpus callosum. Hence, the absence of corpus callosum is a variation which is definitely familial in occurrence and probably inherited as a unit character.

3. The absence of corpus callosum was found only in one line descended from a particular rodless outcross, whereas a sample of eleven mice from the Bagg albino strain, which had a common origin with the rodless strain, showed that it was true-breeding for the normal structure. A number of other inbred strains have been examined, and each was found to be normal.

4. Absence of corpus callosum is undoubtedly not due to the presence of the rodless gene. Of 12 mice with normal rods, 7 lacked the corpus callosum. Of 9 rodless mice, 6 possessed normal brains.

5. There is no sex-linkage. Of 19 males tested, 8 lacked the corpus callosum. Of 26 females, 16 lacked it.

6. The character was found in combination with parted frontals, dominant spotting and silver, suggesting that its determining gene is not closely linked with those underlying the development of the hereditary variations named.

Detailed studies of the genetics, anatomy and physiological reactions of mice lacking the corpus callosum are in progress.

[1] Keeler, C. E., "The Inheritance of a Retinal Abnormality in White Mice," these PROCEEDINGS, **10**, 7, 329–333 (1924).

[2] Keeler, C. E., "Absence héréditaire des bâtonnets chez la souris," *Mem. Soc. Zoöl. France*, **28**, 3 a 5, 48–60 (1927).

[3] Keeler, C. E., Sutcliffe, E., and Chaffee, E. L., "Normal and 'Rodless' Retinæ of the House Mouse with Respect to the Electromotive Force Generated through Stimulation by Light," these PROCEEDINGS, **14**, 6, 477–484 (1928).

[4] Keeler, C. E., "The Geotropic Reaction of Rodless Mice in Light and Darkness," *Jour. Gen. Physiol.*, **11** (4), 361–368 (1928).

[5] Keeler, C. E., "Blind Mice," *Jour. Exp. Zoöl.*, **51** (4), 495–508 (1928).

[6] Keeler, C. E., "Rodless Retina, An Ophthalmic Mutation in the House Mouse, Mus Musculus," *Jour. Exp. Zoöl.*, **46** (4), 355–407 (1927).

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[8] Kappers, C. U. A., *Vergleichende Anatomie des Nervensystems der Wirbeltiere und des Menschen.*, **2**, Haarlem, de Erven F. Bohn (1921).

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[10] Tumbelaka, R., "Das Gehirn eines Affen Worin die Interhemisphäriale Balkenverbindung Fehlt," *Folia Neuro-biologica* **9**, 1 (1915).

A SEARCH FOR EVIDENCE OF THE RADIOACTIVE DECOMPOSITION OF BARIUM

By T. R. Hogness and R. Ruth Comroe

George Herbert Jones Laboratory, University of Chicago

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In 1908 Rayleigh[1] reported the discovery of an exceptionally large helium content in beryl which he had disintegrated and studied spectroscopically for adsorbed gases. Later[2] he suggested that this might be ascribed to the disintegration of the nucleus of the beryllium 8 isotope into two alpha-particles in a manner analogous to the disintegration of radium into radon. Although beryllium is the only element that has given any radioactive evidence that has not been obtained by the usual methods of detection, yet there is no *a priori* reason for supposing it to be unique in this respect. Very slow disintegration accompanied by small energy changes would have escaped detection by the ionization gauge or the photographic plate. The most sensitive test for a very slow change would be one in which an analysis is made for the product of disintegration accumulated over a long time interval, similar to the one Rayleigh inadvertently made with beryllium.

The only elements which could be unambiguously determined as disintegration products rather than "impurities" are the rare gases, and these gases might possibly be alpha-particle disintegration products of the second main group of the periodic system. Of these elements barium offers perhaps the most hopeful possibility since it is the heaviest, and since it, like xenon, undoubtedly consists of many isotopes.[3] If the exact atomic weights of the barium and xenon isotopes were known, it would be possible to calculate thermochemically, at the outset, the possibility of the disintegration:

$$Ba \longrightarrow Xe + He.$$

However, data are lacking for such a calculation.

The radioactivity of barium cannot be completely denied by lack of detectable amounts of xenon in old barium rock, since a half-life period for xenon smaller than that for barium would mean a specific disintegration rate of xenon which would be faster than its formation. The only results, then, which could follow from such an experiment would be either (1) the detection of xenon in old barium compounds which would substantiate the idea of the radioactivity of barium, or (2) the determination, for a hypothetically radioactive barium, of the minimum rate of decay consistent with the failure to detect its activity by the method employed. To this latter conclusion must be added the provision that the barium is disintegrating more rapidly than the xenon.

The experiment was divided into two parts. It was first required to determine how little xenon could be detected spectrographically. This being done it remained to disintegrate very old barium rocks, purify and collect the inert gas evolved, and examine it for traces of xenon.

The method of gas analysis employed followed in general the procedure of Paneth.[4] The apparatus was originally designed to permit the study of very small quantities of xenon in order to determine the lower limit to the amount of the gas which could be detected spectroscopically. In order that the apparatus could be adapted to the final study of the barium rocks, it was so arranged that the gas evolved could be collected in the smallest possible volume.

In general, the apparatus consisted of a thick quartz cell in which the rock was disintegrated, a cell containing calcium for "cleaning up" all except the rare gases, a Toepler pump, and a McLeod gauge which served the additional purpose of compressing the gas into a small definite volume.

A suitably small sample of pure xenon was obtained by a series of expansions into bulbs attached to the vacuum line, and this was introduced into the tip of the McLeod gauge, where, by means of a high tension coil, an electrodeless discharge was sent through it. The principal lines of the xenon spectrum were observed with a Steinheil short focus glass spectrograph, both photographically and visually. In this way it was found that the lower limit of detection was 10^{-7} cc. of xenon measured at standard conditions of temperature and pressure, or about 3×10^{12} atoms of the gas.

The next step was to examine the gases evolved by the disintegration of the barium minerals. The rocks were dissolved in a 50 mol. per cent mixture of sodium chloride and potassium chloride. This mixture was found to be very satisfactory for this purpose. At least 6 g. of barium sulphate could be disintegrated by solution in 10 g. of this solvent.

After the rock had been dissolved in the chloride mixture, the evolved

gases were heated over calcium at about 800°C. and the residual gas was then introduced into the McLeod gauge. Subsequent examination of the texture of the cooled melt after reprecipitation with water showed that all of the rock had been dissolved.

Five rock samples were investigated in this way:

1. Barite from Langmuir township, Ontario, south of Night Hawk Lake. The deposit cuts Early Precambrian (Keewatin rocks) and is supposed to be related in age to Algonian intrusives which came during Early Precambrian time.

2. Barite from the Perry mines in Maddock, Ontario. The vein cuts the Hastings Series and probably dates from Early Paleozoic time.

3. Barite from Claim T. B. 7760 on the north shore of Middle Shebandowan Lake, Ontario. The vein cuts Early Precambrian, and the deposit is inferred to be late Early Precambrian.

4. Barite from Tionoga, Ontario. The deposit is associated with a fluorite bearing pegmatite of the Algoman (Early Precambrian) Age.

5. Hyalophane, a barium feldspar, from Franklin Furnace, New Jersey, which according to Professor Palache of Harvard University, is very probably of great age.

It will be noted that all of the samples examined probably date from the Paleozoic Age, several from its very beginning in Precambrian time. The age of the rocks is apparent from the general assumption that the Paleozoic Age extended approximately from five hundred million to two hundred million years ago.

In each experiment there was a slight residual amount of gas even after purification with calcium. *In no case* was there any evidence of xenon in this gas. The gas from rock 5 was examined spectrographically as well as spectroscopically. A ninety-minute exposure brought out the mercury lines somewhat more intensely than had visual observation, but there was not the slightest indication of even the most intense xenon lines.

If we assume the age of the rock under consideration to be 10^8 years, which is probably the correct order of magnitude, the barium in any particular sample must have decayed at an average rate of at least a thousand molecules a day for enough xenon to have accumulated to make its detection possible. From these data (and assuming a relatively long half-life for xenon) a rough estimate of the maximum half-life of radioactive barium, which would permit of spectroscopic analysis of the product of disintegration, can be calculated to be 10^{18} years. It can also be calculated that if the rate of decay of the hypothetical barium isotope were the same as that of the xenon formed, the corresponding maximum half-life of the barium which would allow for the detection of xenon, would be 10^{16} years. This is almost one million times greater than the half-life of uranium I, the least active of the whole uranium series.

The authors are indebted to Professor Charles Palache of Harvard University, Professor D. Jerome Fischer of the University of Chicago and Dr. T. L. Tanton of the Department of Mines of the Canadian Geological Survey, through whose efforts the rocks used in this investigation were obtained.

[1] Rayleigh, *Proc. Roy. Soc.*, **80**, 575 (1908).
[2] Rayleigh, *Nature*, **123**, 607 (1929).
[3] See Harkins, *P. R.*, **38**, 1270 (1931).
[4] Paneth, *Z. Physik. Chem.*, **134**, 353 (1928).

THE REACTION OF CHLORINE WITH HYDROGEN

BY W. H. RODEBUSH AND W. C. KLINGELHOEFER

CHEMICAL LABORATORY, UNIVERSITY OF ILLINOIS

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The authors have developed the technique of preparing atomic chlorine in the laboratory. The reaction,

$$Cl + H_2 = HCl + H$$

has been investigated at low pressures. This reaction initiates chains which produce at least ten molecules of hydrogen chloride at 1 mm. pressure. Assuming the reaction,

$$H + Cl_2 = HCl + Cl$$

to be rapid it has been found that the chlorine atom has a reaction probability of the order of 10^{-5} per collision with the hydrogen molecule. From the temperature coefficient of the reaction probability it is possible to calculate a heat of activation of 5500 calories. This is much larger than the activation energy assumed by Rollefson and Lindquist[1] but much smaller than the value calculated by Eyring.[2]

Oxygen as might be expected does not affect the reaction at low pressures. The reaction was ordinarily carried out with very dry gases but the addition of water had no effect. This confirms the work of Bodenstein[3] and contradicts the work of Coehn and Jung.[4] The Nernst chain appears to account fully for the reaction mechanism at low pressures. The details of this research will be published elsewhere.

[1] *J. Am. Chem. Soc.*, **52**, 2793 (1930).
[2] *J. Am. Chem. Soc.*, **53**, 2537 (1931).
[3] Private communication.
[4] *Zeit. physik. Chemie*, **110**, 705 (1924).

AN INVESTIGATION OF INTERMEDIATE COMPOUND FORMATION BY MEANS OF THE RAMAN EFFECT

BY JAMES H. HIBBEN

GEOPHYSICAL LABORATORY, CARNEGIE INSTITUTION OF WASHINGTON

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Pursuant to a previous investigation on compound formation with aldehydes,[1] the Raman spectrum of aldol was determined, as this compound contains both aldehydic and hydroxy groups. The Raman shifts ($\Delta\nu$ in cm.$^{-1}$) obtained were the following: 288, 469, 545, 798, 958, 1156, 1307, 1395, 1450, 1638, 1687, 2925 and 2978. As the wave number 1638 would indicate C = C, the spectrum of croton aldehyde was also determined. This yielded $\Delta\nu$ in cm.$^{-1}$ 256, 461, 549, 787, 928, 1048, 1156, 1311, 1395, 1455, 1640, 1690, 2737, 2925 and 3038. While the two spectra have many similarities, the $\Delta\nu$ 787, 928, 1048, 2737 and 3038 cm.$^{-1}$ for the croton aldehyde differ from the corresponding wave number shifts for aldol beyond any experimental error. If shift 1638 for aldol is in reality due to the presence of croton aldehyde as an impurity, then one would also expect the strong lines at 2737 and 3038 cm.$^{-1}$ to be present with an intensity proportional to the aldehyde concentration. Lines characteristic of acetaldehyde,[2] namely, 2850 and 1727 cm.$^{-1}$ are missing. From analogy with para-aldehyde and trioxymethylene the line at 1687 should be missing if para-aldol was the principal constituent. It is possible that aldol may be tautomeric, the double-bonded form being of the ketol type. The possibility that such tautomerism may be common in compounds of this kind has been indicated by Noyes.[3]

Since it is postulated that zinc chloride will form addition compounds with alcohol, and in view of its functioning as a polymerizing agent,[4] the Raman spectra of zinc chloride in ethyl and methyl alcohols were determined. Anhydrous methyl alcohol alone yields $\Delta\nu$ 1036, 1472, 2841 and 2945 cm.$^{-1}$. Lines 1233 and 1375 cm.$^{-1}$, reported by Ganesan and Venkateswaren,[5] are present but if attributed to excitation by the 4078 Å Hg line in place of 4047 Å and 4358 Å the corresponding shifts would be 1042 and 2950 cm.$^{-1}$, which is fairly close to the experimental error. A solution of zinc chloride in an approximate molar ratio of 1:2 gave the following result: the 1036 cm.$^{-1}$ was shifted toward a lower value by 22 wave numbers and the 2841, 2945 cm.$^{-1}$ lines were shifted to a higher value by 23 wave numbers. Shift 1472 cm.$^{-1}$ was greatly reduced in intensity. A time of exposure that brought out the characteristic shift for zinc chloride in aqueous solutions or in ethyl alcohol with great strength yielded no line for zinc chloride. This marked reduction in intensity or suppression of characteristic Raman lines is in contradistinction

to relatively small displacements of 22 wave numbers or less of the lines occasionally brought about by marked changes in electrical environment. A complete disappearance usually indicates a profound alteration in chemical constitution, as for example, in the formation of methylene glycol from formaldehyde and water. Here the typical C = O vibration of the aldehyde is completely suppressed. A somewhat similar behavior in the case of methyl alcohol–zinc chloride mixture may imply a modification of a degree of freedom of the molecule by steric hindrance of the transverse vibration as a result of a coupling of the type postulated by Sidgwick.[6] If oxygen can form a coördinate covalent linkage with a hydrogen atom whose electron is already shared, it seems not wholly irrational to assume that chlorine may do likewise. On the other hand, the effect of the interaction between the zinc chloride and the alcohol may be to make less probable the normal transitions which give rise to particular Raman lines. It is believed, therefore, that the above is evidence of compound formation in solution between zinc chloride and methyl alcohol. The failure of the zinc chloride lines to appear is probably due to a shift toward the lower wave numbers of such magnitude that the Raman frequency cannot be differentiated from the exciting radiation, or they may be so broadened that they are indistinguishable from the general background scattering. The 2841 and 2945 wave numbers correspond to free oscillation of the hydrogen atom against the carbon. The increase in this frequency is probably due to an increase in the force constant, and the modification of the 1036 cm.$^{-1}$ in the opposite direction is probably due to a diminution in the force constant or an increase in the reduced mass. The 1036 cm.$^{-1}$ line is attributed to the HO $\longleftrightarrow$ C oscillation by Trumpy and by Venkateswaren and Bhagavantam.[7] If the zinc chloride forms an addition compound through the hydroxyl group or acts as a dehydration agent, it is remarkable that the 1036 wave number is not more greatly altered. On the contrary, it is the 1472 wave number which suffers the greatest alteration, and this corresponds, as has already been mentioned, to the transverse vibration of the hydrogen atom.[8] The methyl alcohol solution shows a strong continuous spectrum similar to all viscous liquids. This, however, can be considerably diminished by the addition of a fractional per cent of nitro benzene or potassium iodide indicative of de-activation by collisions of the second kind[9] (Klein-Rosseland). Nevertheless, the solution shows no visible absorption, and alcohol alone has no fluorescence of this order of magnitude. Furthermore, while an increase in viscosity would tend to increase fluorescence, increased concentration beyond an optimum amount would act in the opposite direction.[10]

Anhydrous ethyl alcohol gives $\Delta\nu$ in cm.$^{-1}$ of 431, 882, 1049, 1098, 1277, 1457, 2876, 2928 and 2972. No line at 1360 cm.$^{-1}$ was visible.

In an approximate 1:2 solution of $ZnCl_2$ in ethyl alcohol, a strong line for zinc chloride appears at 300 cm.$^{-1}$ and a much weaker component at 308. The 1049, 1098 and 1277 wave numbers show increased diffuseness, and continuous spectrum is apparent. There is no evidence of any marked shift or suppression of any lines in contradistinction to the methyl alcohol example.

Since aluminum chloride is an exceptionally excellent polymerizing agent and since its constitution in solution is open to dispute,[11] it seemed pertinent to ascertain what light, if any, the Raman effect method would throw on this controversy. The Raman spectrum of powdered anhydrous aluminum chloride was therefore determined. This yields a fairly strong line at 541 cm.$^{-1}$ and a weak line at 806 cm.$^{-1}$. A frequency shift less than 350 cm.$^{-1}$ was impossible of determination owing to scattering from the exciting radiation. Both lines are elicited with difficulty, owing to the polar characteristics of the compound.[12] The Raman spectrum of an aqueous solution of aluminum chloride was then determined, which gave $\Delta\nu$ of 348, 541 and 808 cm.$^{-1}$, the latter two having the same relative intensity as the anhydrous material. As there is no reason to assume that gaseous aluminum chloride, having the polymerized form Al_2Cl_6 at temperatures slightly above the boiling point, would depolymerize on solidifying, the presumption is that the aqueous solution contains aluminum chloride in this form. With a force constant of 4.65×10^5 dynes per centimeter, calculated wave numbers 549 and 345 are obtained, respectively, if the assumption is made that the oscillator yielding the former wave number is due to a chlorine atom vibrating linearly against an $AlCl_2$ group and the oscillator yielding the latter wave number is one $AlCl_3$ group oscillating against the other. Presumably this molecule is of the pyramid type, however, so that so simple an agreement may be entirely fortuitous. The water bands which are generally broad and diffuse become stronger and more sharpened than in any solution hitherto examined. The center of this band is at 3418 cm.$^{-1}$, pointing to a modification of the free H atom vibration and corresponding closely to the water band in crystal hydrates.[13] Aluminum chloride solutions of nitro benzene, benzoyl chloride and ethyl alcohol were also examined. The first two gave highly colored products so that it was impossible to determine any effect. The alcoholic solution was slightly colored. The 882 wave number shift was discernible. A strong continuous spectrum tended to mask any lines. This spectrum was present whether excitation was by He 3888 Å, 4471 Å or Hg 4047 Å or 4358 Å lines. Exposures of sufficient duration using every known filtering device failed to bring out any lines for Al_2Cl_6 or any other lines for alcohol except possibly three broad diffuse bands at 4664, 4718 and 4788 Å of uncertain origin. This would seem to indicate a repression of the normal vibrations.

The hydrate of aluminum chloride was also examined. No lines could be elicited under the conditions employed. With the same experimental disposition lines are easily obtained from nitrates, sulphates, bromates or anhydrous aluminum chloride. With such weak Raman lines the general scattering from the arc for the necessarily long exposures tends to mask the plate completely. The same difficulty was encountered with Al_2O_3 as small crystals, one crystal of oriental topaz, and as the precipitate from aluminum chloride solution by means of ammonium hydroxide. In the case of corundum two lines, 931 and 1191 cm.$^{-1}$, appear possible but may be spurious, and two others, 536 and 728 cm.$^{-1}$, if excited by the 4358 Å Hg line, are still less probable. The absence of Raman shifts in corundum would be confirmatory of the ionic nature of the crystal lattice.[14] If the water of crystallization in aluminum chloride hydrate is of the Werner coördination type so that the aluminum is surrounded by water molecules, as has been shown to be the structure of some other chlorides,[15] then it is not surprising that the shifts for anhydrous aluminum chloride are lacking.

The existence of some complexes in solutions has recently been demonstrated by Krishnamurti,[16] by Braune and Engelbrecht[17] and by Bose and Data,[18] by means of the Raman effect. From a somewhat similar point of view the Raman effect in solutions of potassium alum was determined at different concentrations and temperatures. The Raman lines obtained were 461, 609, 795, 982, 1085 and 1160 cm.$^{-1}$. These are in substantial agreement with previous investigators, as characteristic lines of the SO_4 ion, except the lines 795 and 1160. These are weak and diffuse. No line at 2248 reported by Hollaender and Williams[19] was visible. Determination of the Raman shifts for the individual components of potassium alum showed no differences within 3 to 5 wave numbers from that of the potassium alum itself. There is an increased diffuseness of lines, with one exception, with increased temperature, that is, between 25°, 70° and 90°C. A potassium alum glass was prepared by evaporating the solution at 110°C. This showed essentially no crystal structure in its x-ray pattern. The Raman lines were very broad and diffuse but there was no evidence of any change from the normal position. The 982 line remains strong and fairly sharp throughout all these changes. Within the error of observation, therefore, there is no evidence of any compound formation between potassium sulphate and aluminum sulphate in solution; or at least if there is any compound so formed, it does not appreciably alter the Raman spectra of the components.

Aqueous solutions of zinc chloride at different dilutions were next investigated. The exposures were increased in proportion to dilution except for the $^1/_{10}$ molal solution. This gave the following result:

MOLAL CONCENTRATION ZINC CHLORIDE	FREQUENCY SHIFT IN WAVE NUMBERS			ESTIMATED RATIO OF INTENSITY OF 297 TO 396 SHIFT	WATER BAND
11.15	304	402		6:1	Sharp, strong
5.08	297	396		6:1	Sharp, strong
2.40	297	396		6:1	Broadened
1.11	297	396		5:1	Broad
0.558	297	396–510	765?	1:1	Broad, diffuse
0.111*	?	396	765		Broad, diffuse

* One-half the proportionate exposure.

These plates were made with and without a slight excess of HCl. The concentrated solutions showed a strong continuous spectrum characteristic of viscous solutions. There is seemingly an obvious change in the relative intensities of the zinc chloride lines and the appearance of a new band on dilution. This type of effect has been noted by Rao[20] in the case of HNO_3 and attributed by him to a change in ionization. It has already been pointed out by other investigators[21] that zinc chloride ionizes in more than one way. It is presumed that in the less concentrated solutions there exist $ZnCl^+$, Cl^-, Zn^{++} ions and undissociated $ZnCl_2$. Neither Zn^{++} nor Cl^- ions alone will give rise to Raman lines. Therefore the appearance of any new lines must be attributed to a change in the constitution of the solute or solvent. It is in this same concentration region where the new lines appear that previous deviations in ionization have been recorded. Nevertheless, until these experiments can be repeated in a more quantitative manner by means of a microphotometer, results based on estimates of intensity change are to be accepted with some caution. Apparently, however, the following tentative comments may be made. The 297 cm.$^{-1}$ shift, particularly characteristic of un-ionized zinc chloride, remains visible until the concentration of $ZnCl_2$ is less than approximately molal. Indeed, undissociated zinc chloride may still be present in $^1/_{10}$th molal solutions, but owing to the amount of dissociation its actual concentration as zinc chloride may be considerably less. As the Raman effect is proportional to the concentration, the lines may be too weak to register. The intensity of the 297 and 396 cm.$^{-1}$ shifts remains roughly proportional to the dilution until the concentration is below 1 molal, after which the intensity falls off more rapidly than is porportional to dilution. Between 11 and 1 molal solutions there is a distinct sharpening of the zinc chloride lines and a broadening of the water band. In the $^1/_2$ molal solution there appears a faint broad band extending from 396 to 510 wave numbers slightly more intense on the edges. At the same time the relative intensity of the 297 cm.$^{-1}$ line as compared with the 396 cm.$^{-1}$ has decreased. If one calculates the frequency of Cl oscillating against Zn from the assumption that the 296 cm.$^{-1}$ line is due to the zinc oscillating against both Cl atoms, as a first approxima-

tion, the calculated wave number is 365 cm.$^{-1}$. The relative increase in intensity of this line in the dilute solutions may possibly be attributed to the fact that initially ionization diminishes the number of $ZnCl_2$ molecules, but since ZnCl ions are produced, the proportion of these in comparison with $ZnCl_2$ molecules is actually increased. Another line characteristic of zinc chloride may be at 180 cm.$^{-1}$ but has been temporarily rejected from consideration because of a well established slit ghost in the same region.

The Raman spectra of two samples of light gasoline fractions were determined. One of these fractions was highly cracked. Both showed essentially the same spectrum with the exception of a strong line at 1645 cm.$^{-1}$ in the cracked sample. This is characteristic of the olefinic linkage and is consistent with the known olefine content of cracked gasoline. The Raman lines were elicited with greater ease from these samples than from any substance hitherto examined, an exposure of three minutes giving measurable lines. The remaining $\Delta\nu$ in cm.$^{-1}$ were 1005, 1194, 1297, 1458, 2857 and 2917.

The wave number shifts given in this communication are corrected to vacuum. The exciting radiations used were the 4047 Å and the 4358 Å Hg lines, each line being used separately by means of filters to avoid confusion. The spectrograph dispersion was approximately 50 Å per centimeter at 4358 Å. Nearly one hundred fifty plates were taken in the course of this investigation.

Summary.—The Raman spectra for a number of organic and inorganic materials have been given. The evidence so far obtained would indicate possible compound formation in solution between aluminum chloride and ethyl alcohol, polymerization of aluminum chloride in aqueous solution, compound formation in solution between zinc chloride and methyl alcohol and no compound formation in solution between K_2SO_4 and $Al_2(SO_4)_3$. The chemistry of the aldol croton aldehyde type of transformation is indicated as being not so simple as generally supposed. Confirmatory evidence is presented that the ionization of zinc chloride in aqueous solution is a stepwise process. The failure to obtain definite Raman lines from corundum or hydrated aluminum chloride is consistent with the probable character of their lattice.

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DOUBLE INNERVATION OF MELANOPHORES

BY SYLVIA M. MILLS

ZOÖLOGICAL LABORATORY, RADCLIFFE COLLEGE

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There has been much discussion as to whether the movements of fish melanophores are controlled by fibres from the sympathetic system only, or whether the activities of these nerves in producing melanophore contraction are antagonized by those of other nerves which cause expansion. Most of the evidence on this point has been obtained by the use of autonomic drugs. Though the specific action of these substances is too uncertain for the results obtained by their application to be used as final proof, the probability of a double innervation has been greatly increased by recent work in this field. The present paper gives a preliminary description of some experiments with Fundulus heteroclitus which seem to give more direct evidence for a double innervation than has hitherto been presented.

If there exist nerves controlling melanophore expansion as well as melanophore contraction, some differences in the arrangement and distribution of their fibres might be anticipated. It has frequently been shown that a group of denervated melanophores will not *as a whole* respond to changes in the shade of their background, but no attempt has previously been made to show whether precisely the same melanophores fail to contract in response to a white background as fail to expand over black. In investigating this point, an area in the tail fin of Fundulus was denervated as described by Wyman (1924). The fish was kept over a white

background to induce contraction of the melanophores which had expanded as a result of the cut. It was then put over a black background for five minutes to allow the innervated cells to expand. The cells denervated by the cut remained unaltered, and their exact condition and position were recorded. The fish was replaced on a white background, and the tail was again examined after five minutes. The position of the uncontracted melanophores was compared with that of the cells which failed to expand over black. Frequently a few of the *melanophores which expand over a black background fail to contract over white.* In other words, the areas denervated with respect to contraction and to expansion do not coincide melanophore for melanophore. Similar results were obtained when the reactions of innervated melanophores were induced by electrical stimulation. The medulla of a fish with a denervated area in the tail was stimulated with a weak tetanizing current. The innervated melanophores contacted, while the area shown to be denervated by its inability to react to a black background did not usually respond. It frequently happened, however, that more melanophores were left uninfluenced by the current than failed to expand in response to a black background.

When denervating cuts were made in the muscular region at the base of the tail, the behavior of the melanophores in the segment of the fin distal to the cut varied. Most commonly the appropriate stimulations resulted in incomplete contraction and expansion. This indicated that some of both types of nerve fibres supplying the melanophores of this area were cut. In other cases, the melanophores contracted slightly with electrical stimulation, but did not expand when the current was removed, as did normally innervated cells. Conversely, groups of melanophores were sometimes seen which remained unchanged during electrical stimulation, but which expanded further when the current was turned off. These results can readily be interpreted on the basis of a double innervation with melanophore-contracting and -expanding fibres which differ slightly in their distributions. When longitudinal cuts were made parallel to the fin rays, at various levels in the tail, the reactions of melanophores in response to background shade and to electrical stimulations threw further light on the distribution of their expanding and contracting nerves. The results showed that behind the last row of scales, melanophore-contracting fibres run parallel to the fin rays without spreading on either side. The majority of melanophore-expanding fibres also run parallel to the rays, but lateral branches supply a few cells in the immediately adjacent radials or interradials.

Some evidence for the existence of melanophore-expanding nerves has been given. Mechanical stimulation of these nerves may be responsible for the rapid and often maximal expansion of the color cells which follows the section of their nerves. If this be the case, it should be possible to

get subsequent expansion by mechanical stimulation of the distal ends of the cut nerves. A transverse incision was made in the tail fin of Fundulus as before, and the fish was left on a white background to allow these cells to contract. When the knife was again inserted into the cut, and pressed toward its distal face, the cells in the segment of the tail radially subtended from the cut again expanded. This expansion was maintained for three or four hours, but was not so complete as that caused by the original incision. There can be little doubt that this expansion was controlled by nerves. It affected the whole length of the cut fin rays that were restimulated, and the entire width of the denervated area was not necessarily affected. Expansion following mechanical stimulation was not confined to melanophores whose central nervous connections had been severed. Pressure exerted with some blunt instrument such as a glass rod was sufficient to evoke a response in the segment of tail distal to the point of contact. The cells expanded in exactly the same way as if a cut had been made.

Thus the existence of a double innervation of fish melanophores has been demonstrated (*a*) by slight differences in the distribution of the melanophore-contracting and -expanding fibres and (*b*) by the response of melanophores to mechanical stimulation.

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NEURO-HUMORAL CONTROL OF FISH MELANOPHORES

By Sylvia M. Mills

Zoölogical Laboratory, Radcliffe College

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Parker (1932) put forward a neuro-humoral theory of chromatophore control to explain the behavior of both innervated and denervated color cells in fishes and other animals. He suggested that the normal nervous control of chromatophores might be induced by hormones secreted from the terminals of chromatophore nerves in such close proximity to the color cells as to cause them to react. A gradual spread of these substances to the denervated melanophores would explain the response of these cells to changes in the shade of their background. In view of the evidence for a double innervation of the melanophores, both melanophore-contracting and -expanding secretions may be postulated (Mills, 1932). In the present paper, a preliminary account is given of some experiments with Fundulus heteroclitus designed to test this neuro-humoral theory of melanophore control.

Fish melanophores expand immediately that their nerves are cut. The expansion passes off very gradually in 24 to 48 hours and the denervated melanophores will then slowly follow the reactions of the innervated cells. A group of nerves in the tail fin of Fundulus was cut in the way described by Wyman (1924); and the fish was kept on a white background. When the contraction of these recently denervated cells was watched, it was found that they did not contract simultaneously, but those on the edge of the melanophore-expanded region were the first to react. The contraction spread gradually from the periphery of the denervated area, inward. The central cells were thus the last to be affected. According to the neuro-humoral theory, when the fish was on a white background, a melanophore-contracting material was secreted by the melanophore nerve endings. It was being formed on all sides of the denervated area, so that in spreading through the tissues it would first reach and influence the melanophores on the periphery of this area. The more central cells, being further away from the source of secretion, would receive it more slowly and in greater dilution. The general pallor which sometimes follows as a result of shock affects the innervated and denervated cells rapidly and evenly. The reaction in the denervated area is very different from the gradually transmitted response obtained while the fish was over a white background. Since shock pallor is supposedly due to a secretion such as adrenalin formed in a distinct gland and distributed in the blood, it is unlikely that the circulation is also directly responsible for the conduction of the background response.

Any subsequent change in shade that the fish may undergo is transmitted in the same way to the cells of the denervated area. Thus, after about an hour's exposure to a black background, a gradation was seen from the expanded cells on the periphery of the denervated area to the more contracted cells in its center, while the rest of the fish darkened completely. If, while its denervated cells were still in the process of expansion, the fish was transferred from the black background to a white one, the expansion of cells in the denervated area often continued to spread inward, though the surrounding cells were contracting. Similar examples of delayed response have also been observed when the shade of the background was once more changed from black to white.

Melanophores respond more rapidly and completely to electrical and mechanical stimulations of their nerves than they do to changes in shade of their background. This may be due to a greater production of melanophore-motor secretions as a result of the former than of the latter excitations. If this be the case, artificial stimulations may also evoke more rapid responses of denervated cells than do the natural reflexes. When a fish with a denervated area of contracted melanophores was pithed, the innervated cells started to expand immediately, while the denervated

cells were at first unaltered. However, after two or three minutes, expansion began with the cells on the edge of the denervated area, and gradually proceeded inward. Further reactions were complicated by shock pallor. However, when immediately after pithing the tail was severed from the body by a cut just posterior to the anal fin, the innervated cells expanded rapidly and completely, and remained in this condition for several hours. Within two minutes of the operation some of the denervated cells had expanded noticeably though they continued to expand gradually for a period of half an hour to an hour, only those cells on the edge of the denervated area ever reached the same degree of expansion as the adjacent innervated cells. Thus, denervated cells which have remained unaltered after ten minutes over a black background will start expanding in one and a half minutes after central nervous destruction. Since this expansion can be produced in an isolated tail, it obviously is not caused by secretions from glands situated anterior to the anal fin. The way in which the expansion of denervated cells spreads from the periphery to the center of the area is also consistent with the neurohumoral theory of melanophore control.

Parallel results were obtained with electrical stimulation of the medulla. The innervated cells started to contract immediately the current was turned on. After a lag of five to ten minutes they were usually followed in this reaction by the denervated cells. Continued stimulation did not bring the denervated cells to a punctate condition, and after fifteen to twenty minutes they showed no further change. When the current was turned off, the innervated cells began immediately to *expand*, while the denervated cells proceeded to *contract* further. After some minutes, the cells on the periphery of the denervated area again expanded, thus following the reactions of the surrounding innervated cells. When the current was turned on, the expansion of the denervated cells was accelerated, while the innervated cells naturally contracted. The accelerated reactions of the denervated cells, which occur when the current was turned on or off, are possibly correlated with the influence of the current on the vasomotor muscles. A circulation of blood is visible for about half a minute after the application or removal of the stimulus and its influence on the movement of lymph may be responsible for the more rapid spread of melanophore-motor secretions.

Stimulation of the spinal cord in an isolated tail produced the same melanophore reactions as did medullary stimulation of an entire fish. It is therefore not possible for either contractions or expansions of denervated melanophores to be induced by secretions from glands situated anterior to the anal fin. The observations are simply explained if melanophore nerves, when stimulated, produce a secretion causing melanophore

contraction. A distribution of this substance into the surrounding tissue would also account for the observed reactions of denervated cells.

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GAMMA RADIATION

BY ARTHUR BRAMLEY

BARTOL RESEARCH FOUNDATION OF THE FRANKLIN INSTITUTE

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According to the scheme of α energy levels of the radioactive elements the γ radiation should fall into two classes: (1) the lines emitted by one, two or more α particles falling to the ground state from the first excited state and (2) the radiation emitted by an α particle falling from a higher excited state. In the first case the energy of the quanta should be an integral multiple of the energy of the softest ray.[1] The quantum theory of the nucleus is not sufficiently explicit to suggest definitely what should be the law for the second case. It is of interest to inquire how far an empirical relationship suggested by the behavior of the extra nuclear electrons could be developed to suggest the form for the latter case. We should expect that the ratio of the energy of any line to that of the softest should be proportional to the square of the integers.[2] The following table shows that this law holds approximately for the γ rays other than those obeying the linear law.

ELEMENT	HARDEST	SOFTEST	RATIO
RaC′	22.19×10^5	0.59×10^5	37.6
RaB	4.71	0.536	8.76
RaD	0.472	0.472	1.00
$MsTh_2$	9.70	0.581	16.7
ThB	3.02	1.91	1.58*
ThC″	26.49	0.408	64.8
AcC	4.80	3.54	1.36*
Ra	1.89	1.89	1.00
ProtAc	3.23	0.949	3.40
RdAc	3.00	0.315	9.52
AcX	2.69	1.435	1.87*
RdTh	0.881	0.848	1.04

* These lines are possible components of a multiplet with the same n value.

This energy level scheme was arrived at from a consideration of the wave equation, expressed in polar coördinates,

$$\left(1 + \frac{A}{r} + \frac{B}{r^2}\right)\nabla^2\psi + \frac{8\pi^2 m}{h^2}\left(E + \frac{Zze^2}{r}\right)\psi + \left(\frac{Z^2\alpha^2}{r^2} + \frac{\Delta}{r^3}\right)\psi = 0$$

where $$A = -Zh(i)/\sqrt{2\pi^2 m}$$

and α is Sommerfeld's fine structure constant.

It leads to the correct first approximation energy levels for an optical electron, $z = 1$, under the attraction of a central charge Ze. It further leads to a system of energy levels for the nucleus such as we have just been considering.

The terms in A and B in the equation applicable to the nucleus arise whenever the motion of one entity around the center of gravity of a system of entities is considered, all the entities being approximately of the same mass and charge. These terms become of increasing importance as the distances involved diminish. For an electron under the influence of a central charge Ze, the constant (i) should be so chosen that $A = -e^2/4mc^2$.

For the case of a single electron, the constant Δ is very nearly equal to the spin energy term appearing in the usual Hamiltonian, namely, $\frac{Ze^2}{2m^2c^2}\cdot\frac{(L.S)}{r^3}$. For the electronic case, the term in $1/r^3$ may be regarded as arising from the spin interaction term. When, however, we are treating the case of an α particle in the nucleus, a term of the same character must be introduced. This does not necessarily mean that this term arises in the same way in both cases.

The theory of the nucleus has shown the necessity of considering the higher derivatives of the potential, when treating nuclear problems, on account of the small distances involved in these considerations. A similar situation arises when we consider the effect of very hard γ radiation on a free electron. There the higher order terms also become of importance on account of the short wave-length of the radiation.

Equation (1) leads to a type of solution which we can crudely interpret as follows: that the forces acting on an electron approaching the nucleus are first attractive, then as it approaches nearer repulsive and finally attractive again; for a positively charged particle the forces are first repulsive and finally attractive. Thus all particles independent of the sign of their charge are attracted toward it at very small distances from the center of the nucleus and at slightly greater distances repelled. The attractive forces play the predominating part when the distances are so small that the higher order terms become the most important. It is this similarity in the forces between different types of particles in the nucleus

that makes it plausible, for example, to calculate the possibility of an α particle or β particle leaving the nucleus, by the same mechanism.

In the treatment of the perturbation of a free electron by very hard γ rays, the terms in A and B do not occur but the terms arising from the spin interaction must still be retained. We shall do this by replacing the term in Δ/r^3 which is correct for a central field, by a term proportional to $([E \times p].S)$, where E is the electric intensity of the light wave, p the momentum of the electron and S its spin vector. The introduction of the term involving the electric intensity into the equation for the free electron, causes the amount of the Compton scattering to increase or decrease according to the orientation of the spin vector. Since the orientation can have either sign the resulting scattering should show an increase relative to the number of such occurrences as would be calculated from the theory, valid at lower frequencies. This increase is essentially the same as that shown by the Klein-Nishina formula as compared to that of Dirac's.

For the treatment of the photoelectric effect of very hard γ rays, i.e., $h\nu \sim 10^9$ el. volts, on the constituents of the nucleus, the coefficients A and B must be introduced again. We can see, in an approximate manner from equation (1) what might occur. The introduction of the terms A and B has the same effect as decreasing the mass of the particle. If we take the form of the Klein-Nishina coefficient S valid for very hard γ rays,

$$S = \frac{2\pi e^4 \lambda}{hmc^3} [1/4 + \log (1 + 2\alpha)] \quad \alpha = \frac{h\nu}{mc}$$

and replace

$$1/m \text{ by } \left(1 + \frac{A}{r} + \frac{B}{r^2}\right)/m$$

then we should expect that the formula would more nearly approximate the values for the absorption for high frequencies. Introducing this correction and taking the average value of $1/r$ as 1×10^{12}, we obtain for S the expression

$$S = \frac{2\pi e^4 \lambda}{hmc^3} (1 + (i)Z/20 + (i)^2Z^2/500) [1/4 + \log (1 + 2\alpha]$$

where (i) is an integer of the order of unity.

This expression gives a value for the absorption, arising from the nuclear electrons, which is much greater than that for the extra nuclear electrons. The introduction of the factor, containing the coefficients A and B, into the equation for the calculation of the absorption coefficient only has any validity for the closely bound electrons.

The introduction of the term $\left(1 + \frac{A}{r} + \frac{B}{r^2}\right)$ into the ψ equation for

the nuclear electrons has the same effect as reducing the effective mass of the particle. For electrons this loss of mass is a considerable fraction of the total rest mass but for a proton or α particle it is inappreciable. It would seem that, for the distribution of potential near the nucleus, the electron has two positions of equilibrium: (1) where the electron is in an optical level and has lost a negligible amount of its mass and (2) where the electron is within the nucleus and has transformed the major part of its mass into kinetic or potential energy of itself or some other particle. Since the particles are coupled very closely within the nucleus, this transfer of energy is very liable to occur. It is possibly this transformation of mass of the nuclear electrons which makes their properties so elusive to formulate, e.g., β particle emission, loss of spin and absorption of electromagnetic radiation.

Although this presentation is speculative in character, it is hoped that it may serve some useful purpose in formulating a mechanism of nuclear absorption.

Fowler and Miss Swirles have developed a theory to account for the internal absorption coefficient of γ rays. Fowler[3] treated the case of an α particle in an excited state within the nucleus from which it cannot escape by radiation. It must either give its energy over to one of the extra-nuclear electrons or escape by an α disintegration process. In this case, the absorption coefficient is regarded as infinite because no γ radiation of the particular wave-length corresponding to the energy of excitation of the α particle ever appears as such.

The frequency of occurence of this process can be calculated easily from the theory of variation of parameters, as developed by Born. The probability of this event is proportional to the square of the modulus of the integral b

$$b = \int\int (\psi_n\psi_e)_1 \, V(r_e r_n)(\psi_n\psi_e)_2^* \, dw_n dw_e.$$

The ψ's represent the normalized wave functions of the initial and final states of the α particle, subscript n, and of the electron, subscript e, respectively, omitting their time factor. $V(r_e r_n)$ is the energy of interaction of the extra nuclear electron with the α particle and depends to a first approximation on the distance apart of the two particles.

Fowler assumed that $V(r_e r_n)$ could be represented by the expression $Qe^2\{1/R - 1/r_e\}$ $R = |r_e - r_n|$, i.e., the two particles attract each other inversely as the square of their distance apart. It is improbable that such a law of force holds down to nuclear distances. However the contribution to the integral from the region in the immediate vicinity of the nucleus is negligible, since the probability of the electron being in that region is so small.

For the case where the α particle is in a metastable level, Fowler's

assumption regarding the law of force is probably sufficiently correct to give results which agree with the experimental data. When the calculations are carried out, he obtains values which agree quite satisfactorily with experiment. Thus a force varying inversely as the square of the distance at large distances will account completely for the observed transfer of energy between an α particle in a metastable state and the extra nuclear electron.

Miss Swirles,[4] on the contrary, treated the case where the particle in the excited state could emit γ radiation. In order to compute the effect of the radiation on the extra nuclear electrons she treated the nucleus as an Hertzian doublet of frequency ν which perturbed the surrounding planetary electrons. The frequency of the doublet ν is taken to be the same as the frequency of the γ radiation emitted by the nucleus. The vector potential of the Hertzian doublet with its axis along the axis of z and its moment equal to $f(t)$ was given the values $0,0,{}^1/_{cr}\cdot f'(t - r/c)$, and the scalar potential the value $-\frac{\partial}{\partial z}({}^1/_r . f(t - r/e))$, where $f'(t) = -i\nu f(t)$.

In order to calculate the field of an excited α particle in the region surrounding the nucleus, we must make some assumption concerning the nature of the motion of the particle. A simple assumption is that the α particle describes simple harmonic motion of frequency ν; the position of the particle in its path is given by $z = a \cos \nu t$, where a is the amplitude of the oscillation. Its magnitude should be less than the radius of the nucleus. For low frequencies and large distances from the nucleus, the scalar potential ϕ of the field produced by this oscillator may be taken as a first approximation equal to that of a charge $2e$ and a doublet of frequency ν placed at the origin so that $\phi = \frac{2e}{r} - 2ea\frac{\partial}{\partial z}({}^1/_r \cdot e^{i\nu(t-r/c)})$. The first term gives the type of field considered by Fowler, the second term the field whose consequences were developed by Miss Swirles. There is an objection, however, in the case of a radiating α particle to considering that this first approximation will give accurate results as the radiation flux is concentrated around a definite direction.

The scalar potential ϕ of a simple harmonic oscillator can be no longer represented by the approximate field when the frequency of the oscillation is very high. For the γ rays treated by Miss Swirles ν had a minimum frequency of 9×10^{22}. For this case the velocity of the oscillator is of the same order of magnitude as the velocity of light.

We should not only expect that the frequency of the electromagnetic field should be the same as that of the light quantum but also that the half period of decay of the electromagnetic system should be the same as the mean life of the system calculated quantum dynamically. The rate of radiation of energy by the oscillator is given by the expression

$$\frac{2e^2v^2}{3c^3}\ \frac{(1+{}^1/_6\beta^2)}{(1-\beta^2)^4}.$$

This need not disturb us as we should not expect that the classical oscillator has any other properties except its power to portray the electromagnetic field correctly. This same principle has been used in reconciling the dispersion theory with the quantum theory by the introduction of the virtual oscillator. The value of e^2/m of the virtual oscillator is so chosen that under the action of the atomic forces the rate of radiation is the same as that demanded by the quantum theory. The value of e^2/m is related to the frequency ν_{rs} and the Einstein coefficient B_{rs} associated with the transition from the state r to the state s by the relation

$$e^2/m = \frac{3h\nu_{rs}\,B_{rs}}{\pi}.$$

The calculation of the internal absorption coefficient for γ rays from the model of the nucleus in which the field is represented by a virtual oscillator is quite difficult. Ideally, the value of the scalar and vector potentials of the oscillator should be substituted into the wave equation for the electron. As the terms involve the time explicitly the solution of the problem is for all practical purposes unattainable. However, since the period of the virtual oscillator is so short, it might be permissible to take the time average of the scalar potential arising from the oscillator, namely, $1/r\sqrt{1-\tilde{\beta}^2\sin^2\theta}$ $\tilde{\beta} = \frac{a\nu}{c}\sin\nu t$ neglect the vector potential terms and attempt to solve the simplified Schrodinger equation.

However, we shall content ourselves in this article in obtaining an approximate solution by the use of statistical methods. Let $\rho(r\theta\phi)$ be the probability of finding an electron from the K shell at the position designated by r, θ, ϕ, when the nucleus is in the ground state. We shall assume that the concentration of the field arising from the excited α particle, perturbs this distribution of probability in a manner which can be calculated statistically. Taking the time average of the scalar potential as the only part which is effective, we shall assume that the probability of finding the K electron at the position r, θ, ϕ, can be written in the form

$$\bar{\rho} = k\rho(r\theta\phi)e^{2/Z-2\sqrt{1-\beta^2\sin^2\theta}}$$

where Z is the atomic number. The constant k is to be so chosen that the integral of this expression over all space is unity.

The probability of a light quantum giving its energy over to an electron in the K shell is the integral taken over all space of the product of the three following factors: (1) the probability of an electron being in the element

of volume $r^2dr \sin\theta d\theta d\phi$, (2) the probability of a light quantum passing through this element and (3) the probability of a collision between the light quantum and the electron if the light quantum passes through the element of volume at the same time that the electron is in that element. If we define σ_0 as the effective cross sectional area of a light quantum of frequency ν, then

$$P = \sigma_0 \int \bar{\rho}(r\theta\phi)\bar{q}(\theta\phi)r^2dr \sin\theta d\theta d\phi$$

where $\bar{q}$ is the time average of the Poynting flux at $\theta\ \phi$.

Carrying out the integration for the case of a K electron we find that

$$P = {}^3/_7 \left(\frac{Z}{a}\right)^2 e^{\frac{2}{Z-2}/\sqrt{1-\beta^2}} . \sigma_0$$

where $\sigma_0 = 6\alpha \times 10^{-25}$.

For a K electron with a mean life of $1.2 \times 10^{-15}\sqrt{1-\beta^2} = {}^1/_{200}$ and $P = 8.4\ \alpha \times 10^{-3}$.

The expression for P depends on the wave-length of the γ radiation in the same way as σ_0. For γ rays $\alpha \simeq {}^1/_6$ so that

$$P = 1.4 \times 10^{-3}.$$

This simple model of needle radiation gives a value for the internal conversion coefficient for γ rays which is in fair agreement with the observed values for hard γ radiation.

If we carry out the same calculation but assume the simple dipole model for the oscillator then we find a value for P which is much too small. In this case the electrostatic interaction of the electron and the α particle contribute the major portion to the value of the internal conversion factor. In case we are dealing with a metastable state, the mean life of the oscillator is infinite and the electrostatic interaction alone must account for the ejection of the rays. This latter case has been worked out in detail by Fowler.

The model of the oscillator used in this paper to generate the electromagnetic field has been modified with respect to the model used by Fowler and Swirles in order to account for the needle-like character of the radiation field for very high frequencies.

The above analysis in its simple form is applicable only when the α particle is in a state from which one line alone can be emitted. When the α particle is in a state from which two or more transitions are possible to lower energy states, then each transition must be represented by its own electromagnetic oscillator. Let S_{ij} be the strength of the oscillator, representing the transition from the state i to the state j in which radiation of frequency ν_{ij} can be emitted and β_{ij} the corresponding maximum

velocity associated with this oscillator, then the probability of the ejection of a K electron by a quantum of γ radiation is

$$P_i \simeq 2\left(\frac{Z}{a}\right)^2 \sum_j e^{\frac{2}{Z-2} S_{ij}/\sqrt{1-\beta^2_{ij}}}$$

when the α particle is initially in the state i. The S_{ij}'s are determined from the relative intensity of the lines emitted in transition from that state.

Take the particular case where $S_{12} = S_{02} = {}^1/_2$ and $\beta_{12} = \beta_{02} = \beta_{01}\sqrt{1-\beta^2_{01}} = {}^1/_{200}$ then $P_1/P_2 = 4.3$.

This ratio is nearly the same as that of the internal conversion coefficient of γ rays from RaC. There the values of this coefficient can be divided roughly into two groups whose ratio is 1 to 4.

The extension of the analysis to take into account the dependence of P on the number of transitions possible from a given α particle level will account for the variation of the internal conversion factor from one state to another. Although the accuracy of the agreement cited above is probably accidental, it shows the type of dependence of this factor on the state in which the α particle may be present. Nothing further, however, can be definitely stated until the states of RaC are definitely known.

[1] Rutherford and Ellis, *Proc. Roy. Soc.*, **132**, 667 (1931).
[2] Bramley, *Proc. Nat. Acad. Sci.*, **17**, 579 (1931).
[3] Fowler, *Proc. Roy. Soc.*, **129**, 1 (1930).
[4] B. Swirles, *Ibid.*, **116**, 491 (1927).

AN UPPER OLIGOCENE MAMMALIAN FAUNA FROM SOUTHERN CALIFORNIA

By Chester Stock

Balch Graduate School of the Geological Sciences,
California Institute of Technology

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In a previous paper[1] the writer recorded the presence of upper Eocene land mammals in the Sespe deposits north of the Simi Valley, Ventura County, California. Immediately to the west of the Simi Valley occur the Las Posas Hills, where beds mapped as Sespe form a considerably thinner series of sediments. The chance discovery of fragmentary mammalian remains by Dr. W. S. W. Kew on exposures near the top of this section led to further exploration by field parties of the California Institute with the gratifying result that a large and varied vertebrate assemblage

has now been obtained. This important site, C. I. T. Vert. Pale. Loc. 126, has come to be called the Kew Quarry and will be referred to in future discussions of the occurrence and fauna under that name.

The Tertiary and Pleistocene sediments exposed in the Las Posas Hills or determined by the subsurface geology include the Domengine with overlying deposits assigned tentatively to the Tejon, the Sespe, Temblor, Santa Barbara and San Pedro. The sequence is shown in the columnar section (Fig. 1) which I am permitted to reproduce through courtesy of the Shell Company of California. As seen in this section, the Sespe is demarcated from Eocene marine strata below and from middle Miocene marine strata above by erosion intervals. Moreover, faulting which has affected the Sespe deposits below the fossiliferous horizon precludes opportunity to secure an accurate measure of the distance that separates this zone from the bottom of the Sespe. The fossil-bearing beds strike east and west and rest at a high angle, having a dip to the north of 50° or more at the site of the quarry. The sediments comprise maroon-colored clays and greenish sandstones. Immediately overlying the fossil horizon are cross-bedded sands in which occasionally coarse materials occur. Fragments of the maroon or red shale and pebbles of various rock types, several inches in diameter, were noted in the cross-bedded sands. An igneous rock of basaltic character, presumably related in age to the interbedded basic flow or intrusive sill occurring with the Miocene sediments which overlie the Sespe, passes through Sespe deposits in the vicinity of the horizon in which the fossil vertebrates are found. At one locality this basalt intrusive approached within two or three inches of an occurrence of fossil material.

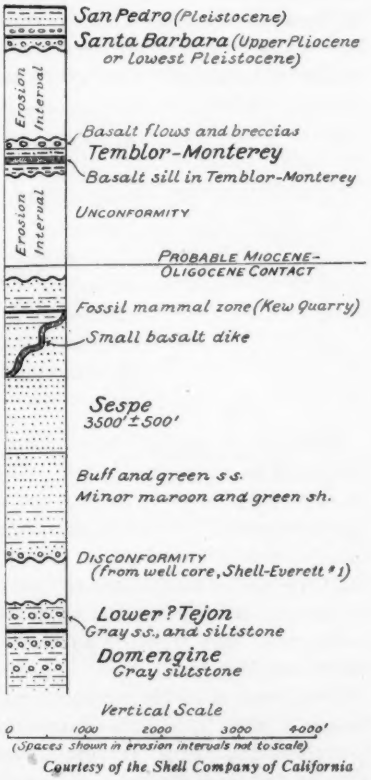

FIGURE 1

Cenozoic stratigraphic column in a section of the Las Posas Hills containing the Kew Quarry fauna, Ventura County, California.

The vertebrate remains were found to be concentrated within a known area of several hundred square feet. Skull and skeletal materials are abundant. Mammalian skulls exhibit particularly the effects of the intense deformation and pressure to which the fossil-bearing beds have been subjected. The specimens are squeezed, crushed and flattened but the teeth, curiously enough, are frequently well preserved. Crushing of the fossil remains may have occurred in part at least during the actual period of accumulation of the deposit.

The following mammals have been identified:

Subhyracodon
Miohippus
Camelid
Hypertragulus
Nimravus
Hoplophoneus
Mesocyon
Temnocyon
Pseudocynodictis?
Ischyromyid rodent
Palaeolagus?

Remains of large tortoises have also been found. Doubtless other forms will be added to this list. While a study of the specific relationships of individual types comprising the assemblage is now in progress, certain age relations of the fauna are already in evidence. The assemblage is clearly more advanced than that recorded from the Sespe north of the Simi Valley. The evolutionary advance which this fauna marks in the history of mammalian life beyond the stage represented by the Simi fauna may have required an elapse of time greater than that included within the Oligocene.

Vertebrate assemblages from the Californian region most nearly related in age to the Kew Quarry fauna are those recorded from the auriferous gravels of the Sierra Nevada, from Tecuya Canyon, southwestern Kern County and from the Sespe of South Mountain, Ventura County.

Whitney[2] recorded the presence of a rhinoceros and an elothere from Douglas Flat and Chili Gulch in Calaveras County. These mammals were described by Leidy as *Rhinoceros hesperius* and *Elotherium superbus* and compared with forms occurring in the John Day beds of Oregon and in the White River badlands of the Great Plains. More recently the former type has been assigned tentatively to the genus *Diceratherium*. Presumably the beds whence these specimens come antedate the andesitic gravels in which late Miocene or Pliocene mammalian remains are found. Of more certain stratigraphic position is the tortoise, *Stylemys calaverensis*,

described by Sinclair[3] from gravels of the rhyolitic epoch near Vallecito, also in Calaveras County.

The Tecuya fauna[4] includes the genus *Hypertragulus* and a rhinoceros determined as *Caenopus?* or *Diceratherium?* on the basis of very fragmentary materials.

No satisfactory comparison between the Kew Quarry fauna and that from the Sespe of South Mountain[5] can be made at present because of absence of oreodonts at the former locality and because of the dearth of fossil mammals on record from the latter region. While the beds containing *Promerycochoerus* (?) *hesperus* Stock may represent a higher horizon than that of the Kew Quarry, definite establishment of their stratigraphic position with reference to the latter must await the discovery of additional faunal evidence. The leptauchenids, whose stratigraphic

		Ventura Co. California	John Day Basin Oregon	Great Plains Rocky Mountains
Lower Miocene		South Mountain–Sespe (Promerycochoerus? Beds)	Upper John Day	Lower Rosebud
Oligocene	Upper	Las Posas Hills–Sespe (Kew Quarry)	Middle John Day Lower John Day	Leptauchenia Beds
	Middle			Oreodon Beds
	Lower			Titanotherium Beds
Upper Eocene		Simi-Sespe		?Duchesne Uinta C

FIGURE 2

Correlation chart showing time relations of Kew Quarry fauna to some early Tertiary vertebrate assemblages of western North America.

range extends lower in the Sespe section of South Mountain, may occur in deposits broadly equivalent to those containing the Kew Quarry fauna. If this is definitely determined their absence in the latter assemblage may be ascribed to environmental conditions unfavorable to the existence of these creatures at the time and place of accumulation of the organic remains in the Las Posas Hills horizon.

The presence of the rhinoceros *Subhyracodon* and of an ischyromyid rodent suggests a relation with the middle or upper White River Oligocene of the western Great Plains. The absence of titanotheres and of hyaenodonts and the presence of a horse clearly more advanced than *Mesohippus bairdii* indicate that the horizon is certainly of later age than the Titanotherium beds of the lower White River and later than the Oreodon beds of the middle White River. The carnivores and the camelid likewise suggest a later age. Presence of *Miohippus*, *Nimravus*, *Mesocyon* and

Temnocyon points rather strongly to a close bond with the John Day fauna of north-central Oregon.

In contrast to the diceratheres of the John Day, the hornless rhinoceroses of the Las Posas Hills-Sespe represent an earlier stage in the phylogenetic history of this group. Moreover, while the rodents of the John Day comprise a somewhat varied group, members of the family Ischyromyidae are absent. The presence of *Subhyracodon* and an ischyromyid rodent in the Kew Quarry horizon may be interpreted as indicating an age for this zone in the Sespe earlier than that of the John Day. On the other hand, these forms may have persisted for a longer period in the region of southern California than in the John Day basin of Oregon. The assemblage from the John Day deposits is regarded as upper Oligocene in age. In recent years, however, a lower Miocene age has been advocated for the John Day fauna, at least for that portion of the assemblage known from the upper John Day (*Promerycochoerus* beds). In view of the relationships which certain elements of the Kew Quarry fauna have with types occurring in the later White River and in the John Day, the age of this stage of the Sespe can be stated at present as being not later than lower Miocene or earlier than upper Oligocene. The time relations of the fauna may be tentatively expressed in the diagram, figure 2.

[1] Stock, C., *Proc Nat. Acad. Sci.*, **18**, 518–523 (1932).
[2] Whitney, J. D., *Mem. Mus. Comp. Zoöl Harvard*, **6**, 243–245 (1880).
[3] Sinclair, W. J., *Univ. Calif. Publ. Bull. Dept. Geol.*, **3**, 11 (1903).
[4] Stock, C., *Univ. Calif. Publ. Bull. Dept. Geol.*, **12**, 4 (1920).
[5] Stock, C., *Carnegie Inst. Wash. Publ.*, **404**, art. 3 (1930).

AN EXTENSION OF THE NEW EINSTEIN GEOMETRY[1]

BY A. D. MICHAL AND J. L. BOTSFORD

DEPARTMENT OF MATHEMATICS, CALIFORNIA INSTITUTE OF TECHNOLOGY

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The geometry considered by Einstein[2] and Mayer in their "Unified Field Theory" leads to the consideration of an n-dimensional Riemannian space V_n with a metric tensor g_{ij}, to each point of which is associated an m-dimensional linear vector space V_m $(m > n)$, for which vector spaces a general linear connection is defined. For the general case $(m - n \neq 1)$ we find that the calculation of the $m - n$ "exceptional directions" is not unique, and that an additional postulate on the linear connection is necessary. Several of the theorems give new results even for $n = 4$, $m = 5$, the Einstein-Mayer case. It is of interest to note that the dependence of the Einstein-Mayer unified field equations on V_4 tensors is not only a

property peculiar to the tensors involved in the field equations, but is a property common to all equations involving tensors of the form (2). This follows from the corollary to theorem 7 below.

We adopt the convention that small Latin indices range from 1 to n, small Greek indices from 1 to m and capital Latin indices from $n+1$ to m. Indices repeated in the same term are to be summed over their respective ranges.

We shall consider only uniquely reversible transformations of coördinates in V_n, and in V_m only linear transformations with coefficients which are functions of the coördinates x^i, and whose determinants do not vanish. We shall say that a given set of m^2 functions $M^\alpha_\beta(x)$ determine a transformation of vector coördinates from one vector coördinate system to another. The reference system composed of a coördinate system x^i and a vector coördinate system will be called a representation.

The law of transformation of a composite tensor under a change of representation will be taken to be

$$X^{\cdots\alpha\cdots a\cdots}_{\cdots\beta\cdots b\cdots}(x)\cdots M^\beta_\mu\cdots\frac{\partial x^b}{\partial\bar{x}^s}\cdots=\bar{X}^{\cdots\lambda\cdots r\cdots}_{\cdots\mu\cdots s\cdots}(\bar{x})\cdots M^\alpha_\lambda\cdots\frac{\partial x^a}{\partial\bar{x}^r}\,.$$

Let V_m be endowed with a symmetric metric tensor $G_{\alpha\beta}$, where $G_{\alpha\beta}X^\alpha X^\beta$ is positive definite throughout V_n. Further assume the existence of a composite tensor γ^α_a for which $\| \gamma^\alpha_a \|$ is of rank n.

THEOREM 1. There exist $m-n$ linearly independent normal orthogonal (with respect to $G_{\alpha\beta}$) vector fields A^E_α which are solutions of

$$\gamma^\alpha_a X_\alpha = 0.$$

Any two such sets of vector field are related by

$$'A^E_\alpha(x) = L^E_F(x)\, A^F_\alpha(x),$$

where $\| L^E_F \|$ is an orthogonal matrix.

THEOREM 2. The form $G_{\alpha\beta}\,\gamma^\alpha_a\,\gamma^\beta_b\,dx^a\,dx^b$ is positive definite.

We impose the condition that the Riemannian metric is given by

$$g_{ab} = \gamma^\alpha_a\,\gamma^\beta_b\,G_{\alpha\beta}. \tag{1}$$

By means of the fundamental composite tensor γ^α_a we define a correspondence between V_m and V_n vectors by

$$b^a = \gamma^a_\alpha\,B^\alpha$$

$$b_a = \gamma^\alpha_a\,B_\alpha.$$

THEOREM 3. The inverse of (1) is

$$G_{\alpha\beta} = g_{ab}\,\gamma^a_\alpha\,\gamma^b_\beta + A^E_\alpha\,A^E_\beta.$$

A linear connection $\Gamma^{\alpha}_{\beta a}(x)$, i.e., an invariant whose law of transformation is

$$M^{\alpha}_{\sigma}\,\overline{\Gamma}^{\sigma}_{\beta r}(\bar{x}) = \left(\Gamma^{\alpha}_{\lambda a}(x)\,M^{\lambda}_{\beta} + \frac{\partial M^{\alpha}_{\beta}}{\partial x^{a}}\right)\frac{\partial x^{a}}{\partial \bar{x}^{r}},$$

will be defined in terms of the quantities γ^{α}_{a}, A^{α}_{E}, $G_{\alpha\beta}$ and g_{ab}, and a set of arbitrarily chosen V_n tensors F^{E}_{ac}, by means of the following four compatible postulates.

Postulate I. $G_{\alpha\beta;\,c} \equiv \dfrac{\partial G_{\alpha\beta}}{\partial x^{c}} - \Gamma^{\lambda}_{\alpha c}\,G_{\lambda\beta} - \Gamma^{\lambda}_{\beta c}\,G_{\alpha\lambda} = 0.$

Postulate II. $\gamma^{\alpha}_{a;\,c} \equiv \dfrac{\partial \gamma^{\alpha}_{a}}{\partial x^{c}} + \Gamma^{\alpha}_{\lambda c}\,\gamma^{\lambda}_{a} - \left\{{h \atop ac}\right\}\gamma^{\alpha}_{h} = A^{\alpha}_{E}\,F^{E}_{ac},$

where $\left\{{h \atop ac}\right\}$ are the Christoffel symbols of V_n.

Postulate III. $F^{E}_{ac} = -\,F^{E}_{ca}.$

Postulate IV. $A^{\alpha}_{E;\,c}\,A^{F}_{\alpha} = 0.$

Except for the one on V_m geodesics, the theorems presented in this paper do not depend on the skew-symmetry of the tensors F^{E}_{ac}. Hence a more general geometry may be developed on the basis of postulates I, II and IV alone.

It is to be noted that if $m - n = 1$, Postulate IV is not necessary, but is a consequence of definitions and Postulate I.

THEOREM 4. The precise form of the linear connection as determined by the above postulates is given by

$$\Gamma^{\alpha}_{\beta a} = \left(F^{E}_{ca}\,\gamma^{c}_{\beta} + \frac{\partial A^{E}_{\beta}}{\partial x^{a}}\right)A^{\alpha}_{E} + \left(\frac{\partial \gamma_{\beta c}}{\partial x^{a}} - \gamma_{\beta h}\left\{{h \atop ca}\right\} - A^{E}_{\beta}\,F^{E}_{ca}\right)\gamma^{ac}.$$

Several theorems on parallel displacement of vectors have been proved, but will not be given here. From these theorems may be derived an equivalent set of postulates.

THEOREM 5. In the new representation derived from a given representation by the transformation of vector coördinates determined by

$$M^{\alpha}_{a} = \gamma^{\alpha}_{a}$$
$$M^{\alpha}_{A} = A^{\alpha}_{A}$$

the components of the fundamental quantities are given by

$$\overline{\gamma}^{\alpha}_{a} = \delta^{\alpha}_{a},\ \overline{\gamma}^{a}_{\alpha} = \delta^{a}_{\alpha},\ A^{\alpha}_{E} = \delta^{\alpha}_{E},\ A^{E}_{\alpha} = \delta^{E}_{\alpha}$$
$$\overline{G}_{\alpha\beta} = \delta^{a}_{\alpha}\,\delta^{b}_{\beta}\,g_{ab} + \delta^{A}_{\alpha}\,\delta^{A}_{\beta}$$
$$\overline{\Gamma}^{\alpha}_{\beta c} = \delta^{\alpha}_{a}\,\delta^{b}_{\beta}\left\{{a \atop bc}\right\} + \delta^{\alpha}_{A}\,\delta^{b}_{\beta}\,F^{A}_{bc} + \delta^{\alpha}_{a}\,\delta^{B}_{\beta}\,F^{a}_{Bc}.$$

Let $^{\dagger}\Gamma^{\alpha}_{\beta a}$ stand for the components of the connection evaluated in a

representation composed of a Riemannian normal coördinate system y^i with origin $x^i = q^i$ and the vector coördinate system described in theorem 5. Define the functions $\Lambda^\alpha_{\beta a c_1 \ldots c_k}(x)$ by

$$\Lambda^\alpha_{\beta a c_1 \ldots c_k}(q) = \left(\frac{\partial^k \Gamma^\alpha_{\beta a}}{\partial y^{c_1} \ldots \partial y^{c_k}}\right)_0.$$

THEOREM 6. Any tensor differential invariant with no Greek indices

$$T^{i \ldots j}_{k \ldots l}\left(\Gamma^\alpha_{\beta a}, \frac{\partial \Gamma^\alpha_{\beta a}}{\partial x^{c_1}}, \ldots, \frac{\partial^s \Gamma^\alpha_{\beta a}}{\partial x^{c_1} \ldots \partial x^{c_s}}; \ \gamma^\alpha_a, \frac{\partial \gamma^\alpha_a}{\partial x^c}, \ldots; \ A^\alpha_E, \frac{\partial A^\alpha_E}{\partial x^c} \ldots; \right. \tag{2}$$

$$\left. g_{ab}, \frac{\partial g_{ab}}{\partial x^c}, \frac{\partial^2 g_{ab}}{\partial x^{c_1} \partial x^{c_2}}, \ldots, \frac{\partial^t g_{ab}}{\partial x^{c_1} \ldots \partial x^{c_t}}; \ G_{\alpha\beta}, \frac{\partial G_{\alpha\beta}}{\partial x^c}, \frac{\partial^2 G_{\alpha\beta}}{\partial x^{c_1} \partial x^{c_2}}, \ldots, \frac{\partial^u G_{\alpha\beta}}{\partial x^{c_1} \ldots \partial x^{c_u}}\right)$$

may be expressed as the same functions with arguments which are linear combinations of V_n tensors; that is, it may be put in the form

$$T^{i \ldots j}_{k \ldots l}(\Lambda^\alpha_{\beta a}, \Lambda^\alpha_{\beta a c}, \ldots, \Lambda^\alpha_{\beta a c_1 \ldots c_s}; \ \delta^\alpha_a, 0, \ldots; \ \delta^\alpha_E, 0, \ldots;$$

$$g_{ab}, 0, g_{ab,c_1 c_2}, \ldots, g_{ab,c_1 \ldots c_t}; \ \delta^a_\alpha \delta^b_\beta g_{ab} + \delta^A_\alpha \delta^A_\beta, 0,$$

$$\delta^a_\alpha \delta^b_\beta g_{ab,c_1 c_2}, \ldots, \delta^a_\alpha \delta^b_\beta g_{ab,c_1 \ldots c_u})$$

by replacing $\Gamma^\alpha_{\beta a}$ by $\Lambda^\alpha_{\beta a}$, $\frac{\partial^k \Gamma^\alpha_{\beta a}}{\partial x^{c_1} \ldots \partial x^{c_k}}$ by $\Lambda^\alpha_{\beta a, c_1 \ldots c_k}$, γ^α_a by δ^α_a, derivatives of γ^α_a by zero, A^α_E by δ^α_E, derivatives of A^α_E by zero, $\frac{\partial g_{ab}}{\partial x^c}$ by zero, $\frac{\partial^k g_{ab}}{\partial x^{c_1} \ldots \partial x^{c_k}}$ by the k^{th} affine extension of g_{ab}, $G_{\alpha\beta}$ by $\delta^a_\alpha \delta^b_\beta g_{ab} + \delta^A_\alpha \delta^A_\beta$, $\frac{\partial G_{\alpha\beta}}{\partial x^c}$ by zero and $\frac{\partial^k G_{\alpha\beta}}{\partial x^{c_1} \ldots \partial x^{c_k}}$ by $\delta^a_\alpha \delta^b_\beta$ times the k^{th} affine extension of g_{ab}.

THEOREM 7. The components of any tensor differential invariant of the form (2) are expressible as functions of V_n tensors only; i.e.,

$$T^{i \ldots j}_{k \ldots l} = H^{i \ldots j}_{k \ldots l}(A^h_{abc}, \ldots, A^h_{abc_1 \ldots c_s}; \quad F^E_{ab}, F^E_{ab,c}, \ldots, F^E_{ab,c_1 \ldots c_s};$$

$$g_{ab}, 0, g_{ab,c_1 c_2}, \ldots, g_{ab,c_1 \ldots c_q})$$

where $A^h_{abc_1 \ldots c_k}$ is the k^{th} affine normal tensor,[3] $F^E_{ab,c_1 \ldots c_k}$ and $g_{ab,c_1 \ldots c_k}$ are the k^{th} affine extensions of F^E_{ab} and g_{ab}, respectively,[3] and where q is the greatest of s, t and u.

COROLLARY. The tensor differential invariant (2) may be expressed as

$$T^{i \ldots j}_{k \ldots l} = K^{i \ldots j}_{k \ldots l}(K^l_{hab}, K^l_{hab;c_1}, K^l_{hab;c_1 c_2}, \ldots, K^l_{hab;c_1;\ldots;c_w};$$

$$F^E_{ab}, F^E_{ab;c_1}, F^E_{ab;c_1 c_2}, \ldots, F^E_{ab;c_1;\ldots;c_s}; \quad g_{ab})$$

where

(1) K^l_{hab} is the Riemann-Christoffel tensor of V_n,

(2) a semicolon between indices indicates covariant differentiation based on the Christoffel symbols of V_n,

(3) w is the greatest of $s-1$, $t-2$ and $u-2$.

THEOREM 8. The equations determining a V_m geodesic are

$$\frac{d^2x^i}{ds^2} + \left\{ \begin{matrix} i \\ ha \end{matrix} \right\} \frac{dx^h}{ds}\frac{dx^a}{ds} = c^E F^i_{Ea} \quad (F^i_{Ea} \equiv g^{is} F^E_{as})$$

where c^E are constants along the geodesic. The skew-symmetry of the tensors F^E_{as} is involved only in proving the constancy of c^E along the geodesic.

Evident modifications of the formulas and theorems of this paper will result from substituting for Postulate II the more general postulate[4]

$$\gamma^\alpha_{a;\,c} = A^\alpha_E F^E_{ac} + \gamma^\alpha_r U^r_{ac}.$$

[1] Presented to the Amer. Math. Society, March, 1932.

[2] A. Einstein and W. Mayer, *Sitzungberichte der Preuss. Akad.*, 541–557, Dec., 1931. We assume that the reader is familiar with the mathematical content of this paper.

[3] O. Veblen and T. Y. Thomas, *Trans. Amer. Math. Soc.*, **25** (1923); O. Veblen, *Invariants of Quadratic Differential Forms*, Cambridge Tract (1927). In the present paper we shall understand that all affine normal tensors and affine extensions of tensors are based on the Christoffel symbols of V_n.

[4] A. Einstein and W. Mayer, *Sitzungberichte der Preuss. Akad.*, 130–137, April 14, 1932. See the corresponding modifications made for the purpose of treating the more complete unified field equations.

SIMULTANEOUS DIFFERENTIAL INVARIANTS OF AN AFFINE CONNECTION AND A GENERAL LINEAR CONNECTION

BY A. D. MICHAL AND J. L. BOTSFORD

DEPARTMENT OF MATHEMATICS, CALIFORNIA INSTITUTE OF TECHNOLOGY

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Let Γ^i_{jk} be a symmetric affine connection, i.e., an invariant whose transformation law under a transformation of coördinates

$$\bar{x}^j = \bar{x}^j(x^1, \ldots, x^n), \quad \left| \frac{\partial \bar{x}^i}{\partial x^j} \right| \neq 0$$

is

$$\bar{\Gamma}^i_{jk}(\bar{x}) = \left(\Gamma^a_{bc}(x) \frac{\partial x^b}{\partial \bar{x}^j}\frac{\partial x^c}{\partial \bar{x}^k} + \frac{\partial^2 x^j}{\partial \bar{x}^j \partial \bar{x}^k} \right) \frac{\partial \bar{x}^i}{\partial x^a}.$$

Also, consider a general linear connection $\Gamma^\alpha_{\beta a}$, i.e., an invariant whose transformation law under a change of representation[1] is

$$M^{\alpha}_{\sigma}(x)\,\bar{\Gamma}^{\sigma}_{\beta a}(\bar{x}) = \left(\Gamma^{\alpha}_{\lambda\tau}(x)\,M^{\lambda}_{\beta}(x) + \frac{\partial M^{\alpha}_{\beta}(x)}{\partial x^{\tau}}\right)\frac{\partial x^{\tau}}{\partial \bar{x}^{a}}.$$

A change of representation is determined by a transformation of coördinates, and the transformation of vector coördinates

$$V^{\alpha} = M^{\alpha}_{\beta}\,\bar{V}^{\beta}, \quad |M^{\alpha}_{\beta}| \neq 0.$$

In the above formulas, as throughout this paper, we shall understand that small Latin indices range from 1 to n, and small Greek indices from 1 to m. Indices repeated in the same term are to be summed over their respective ranges.

We shall deal with composite tensors, i.e., tensors which involve both Greek and Latin indices. As special cases, we have V_n tensors, i.e., tensors involving only Latin indices (usual tensors), and V_m tensors, i.e., tensors involving only Greek indices. For example, the law of transformation of the composite tensor $X^{\alpha a}_{\beta b}$ is

$$X^{\alpha a}_{\beta b}(x)\,M^{\beta}_{\mu}\frac{\partial x^{b}}{\partial \bar{x}^{s}} = \bar{X}^{\lambda r}_{\mu s}(\bar{x})\,M^{\alpha}_{\lambda}\frac{\partial x^{a}}{\partial \bar{x}^{r}}.$$

Define a set of functions $\Gamma^{\alpha}_{\beta a_1 \cdots a_r}(x)$ as follows:

$$\Gamma^{\alpha}_{\beta ab} = \frac{1}{2}P\left(\frac{\partial \Gamma^{\alpha}_{\beta a}}{\partial x^{b}} - \Gamma^{\alpha}_{\lambda a}\,\Gamma^{\lambda}_{\beta b}\right)$$

$$\Gamma^{\alpha}_{\beta a_1 \cdots a_{r+1}} = \frac{1}{r+1}P\left(\frac{\partial \Gamma^{\alpha}_{\beta a_1 \cdots a_r}}{\partial x^{a_{r+1}}} - \Gamma^{\alpha}_{\lambda a_1 \cdots a_r}\,\Gamma^{\lambda}_{\beta a_{r+1}}\right)$$

where $P(\ldots)$ denotes the sum of all terms derived from the terms in the parenthesis by cyclic permutation of the Latin indices.

THEOREM 1. The law of transformation of the functions $\Gamma^{\alpha}_{\beta a_1 \cdots a_p}$ under a transformation of vector coördinates only is

$$M^{\sigma}_{\beta}\Gamma^{\alpha}_{\sigma a_1 \cdots a_p} = M^{\alpha}_{\lambda}\bar{\Gamma}^{\lambda}_{\beta a_1 \cdots a_p} + C\left[\sum_{k=1}^{p-1}\frac{\partial^{k} M^{\alpha}_{\lambda}}{\partial x^{a_1}\ldots\partial x^{a_k}}\bar{\Gamma}^{\lambda}_{\beta a_{k+1}\cdots a_p}\right] - \frac{\partial^{p} M^{\alpha}_{\beta}}{\partial x^{a_1}\ldots\partial x^{a_p}},$$

where $C[\ldots]$ denotes the sum of all the different terms derived from the terms in the bracket by taking all permutations of Latin indices.

Let the symbol * denote that components of tensors, connections, etc., are referred to an affine normal coördinate system.[3]

THEOREM 2. If the coördinate systems of two representations are affine normal coördinate systems with a common origin $x^{i} = \bar{x}^{i}$, and the vector coördinate systems are related by the coefficients M^{α}_{β}, the functions ${}^{*}\bar{\Gamma}^{\alpha}_{\beta a_1 \cdots a_r}$ and ${}^{*}\bar{\Gamma}^{\alpha}_{\beta a_1 \cdots a_r}$ are related by the equations

$$M^{\alpha}_{\sigma}\,{}^{*}\bar{\Gamma}^{\sigma}_{\beta a_1 \cdots a_r}(\bar{y}) = \left[M^{\lambda}_{\beta}\,{}^{*}\Gamma^{\alpha}_{\lambda s_1 \cdots s_r}(y) + \frac{\partial^{r} M^{\alpha}_{\beta}(y)}{\partial y^{s_1}\ldots\partial y^{s_r}}\right]\frac{\partial y^{s_1}}{\partial \bar{y}^{a_1}}\cdots\frac{\partial y^{s_r}}{\partial \bar{y}^{a_r}} + \{\cdots\}$$

where $\{ \dots \}$ indicates terms which vanish with ${}^*\bar{\Gamma}^{\alpha}_{\beta a_1 \dots a_k}$ $(k < r)$ and higher derivatives than the first of y^s.

We define a geodesic vector coördinate system of order r with the base point $x^i = q^i$, to be a vector coördinate system in which the functions $\Gamma^{\alpha}_{\beta a}(x)$, $\Gamma^{\alpha}_{\beta a_1 a_2}(x), \dots, \Gamma^{\alpha}_{\beta a_1 \dots a_r}(x)$, evaluated at $x^i = q^i$ are zero.[2]

THEOREM 3. If m^{α}_{β} are m^2 numbers such that $|m^{\alpha}_{\beta}| \neq 0$, the new vector coördinate system determined from the old by

$$M^{\alpha}_{\beta}(x) = m^{\lambda}_{\beta}[\delta^{\alpha}_{\lambda} - (\Gamma^{\alpha}_{\lambda a})_q \frac{(x^a - q^a)}{1!} - (\Gamma^{\alpha}_{\lambda ab})_q \frac{(x^a - q^a)(x^b - q^b)}{2!} - \cdots - (\Gamma^{\alpha}_{\lambda a_1 \cdots a_r})_q \frac{(x^{a_1} - q^{a_1}) \cdots (x^{a_r} - q^{a_r})}{r!}$$

is a geodesic vector coördinate system of order r with the base point $x^i = q^i$.

By a *geodesic representation* of order r with the base point $x^i = q^i$ we shall mean a representation derived from a given representation by a transformation of coördinates to affine normal coördinates[3] with origin at $x^i = q^i$, followed by the transformation of vector coördinates determined by

$$R^{\alpha}_{\beta}(y) = \delta^{\alpha}_{\beta} - \left({}^*\Gamma^{\alpha}_{\beta a}(y)\right)_0 y^a - \left({}^*\Gamma^{\alpha}_{\beta ab}(y)\right)_0 \frac{y^a y^b}{2!} - \cdots - \left({}^*\Gamma^{\alpha}_{\beta a_1 \dots a_r}(y)\right)_0 \frac{y^{a_1} \dots y^{a_r}}{r!}.$$

Thus, the vector coördinate system in a geodesic representation of order r with the base point $x^i = q^i$ is a geodesic vector coördinate system of order r with the base point $y^i = 0$.

Let the symbol † denote that the components of tensors, connections, etc., are referred to a geodesic representation of order r.

THEOREM 4. If two representations are related by the transformation of coördinates $\bar{x}^i = \bar{x}^i(x)$, and the transformation of vector coördinates determined by the coefficients M^{α}_{β}, and if we effect changes of representation to two geodesic representations of order r with a common base point $x^i = q^i$, the two geodesic representations are related by the transformation of coördinates $\bar{y}^i = \bar{y}^i(y)$, where

$$\left|\frac{\partial \bar{y}^i}{\partial y^j}\right|_0 \neq 0, \left(\frac{\partial \bar{y}^i}{\partial y^j}\right)_0 = \left(\frac{\partial \bar{x}^i}{\partial x^j}\right)_q, \frac{\partial^s \bar{y}^i}{\partial y^{j_1} \dots \partial y^{j_s}} = 0 \ (s > 1),$$

and the transformation of vector coördinates determined by $T^{\alpha}_{\beta}(y)$, where

$$|T^{\alpha}_{\beta}|_0 \neq 0, (T^{\alpha}_{\beta})_0 = (M^{\alpha}_{\beta}(x))_q, \left(\frac{\partial^s T^{\alpha}_{\beta}}{\partial y^{j_1} \dots \partial y^{j_s}}\right)_0 = 0 (s = 1, 2, \dots, r).$$

THEOREM 5. The quantities $A^{\alpha}_{\beta a c_1 \cdots c_p}(x)$ defined by

$$A^{\alpha}_{\beta a c_1 \cdots c_p}(q) = \left(\frac{\partial^p \, {}^{\dagger}\Gamma^{\alpha}_{\beta a}}{\partial y^{c_1} \ldots \partial y^{c_p}} \right)_0$$

are components of a composite tensor of the indicated rank, to be known as the pth König normal tensor. Besides the obvious symmetries, this tensor satisfies the relations

$$P(A^{\alpha}_{\beta a b_1} \ldots b_r) = 0.$$

THEOREM 6. The quantities $X^{\cdot\alpha a\cdot}_{\cdot\beta b\cdot, c_1 \cdots c_p}(x)$ defined in terms of the composite tensor $X^{\cdot\alpha a\cdot}_{\cdot\beta b\cdot}$ by

$$X^{\cdot\alpha a\cdot}_{\cdot\beta b\cdot, c_1 \cdots c_p}(q) = \left(\frac{\partial^p \, {}^{\dagger}X^{\cdot\alpha a\cdot}_{\cdot\beta b\cdot}}{\partial y^{c_1} \ldots \partial y^{c_p}} \right)_0$$

are the components of a composite tensor of the indicated rank, to be known as the pth extension of the tensor $X^{\cdot\alpha a\cdot}_{\cdot\beta b\cdot}$.

When $X^{\cdot\alpha a\cdot}_{\cdot\beta b\cdot}$ is a V_n tensor, the extension so defined reduces to the affine extension.[3]

We can calculate the explicit form of the König normal tensors and the extensions of tensors. For instance:

THEOREM 7. The first and second König normal tensors are given by

$$A^{\alpha}_{\beta a b} = \tfrac{1}{2} P^{\alpha}_{\beta a b}$$

and

$$A^{\alpha}_{\beta a b c} = \tfrac{1}{3} (P^{\alpha}_{\beta a b; c} + P^{\alpha}_{\beta a c; b}),$$

respectively, where $P^{\alpha}_{\beta a b}$ is the curvature tensor based on the linear connection $\Gamma^{\alpha}_{\beta a}$.

THEOREM 8. The first extension of the composite tensor $X^{\alpha a}_{\beta b}$ is the covariant derivative of $X^{\alpha a}_{\beta b}$, and the second extension is given by

$$X^{\alpha a}_{\beta b, c d} = X^{\alpha a}_{\beta b; c; d} - A^{\alpha}_{\lambda c d} X^{\lambda a}_{\beta b} - A^{a}_{h c d} X^{\alpha h}_{\beta b} + A^{\lambda}_{\beta c d} X^{\alpha a}_{\lambda b} + A^{h}_{b c d} X^{\alpha a}_{\beta h}.$$

THEOREM 9. The second extension of a V_m tensor $X^{\alpha \ldots \beta}_{\lambda \ldots \mu}$ is given by

$$X^{\alpha \ldots \beta}_{\lambda \ldots \mu, c d} = \tfrac{1}{2} [X^{\alpha \ldots \beta}_{\lambda \ldots \mu; c; d} + X^{\alpha \ldots \beta}_{\lambda \ldots \mu; d; c}].$$

THEOREM 10. Any tensor differential invariant of the form

$$T^{\alpha a \ldots}_{\beta b \ldots}\left(\Gamma^{\alpha}_{\beta a}, \frac{\partial \Gamma^{\alpha}_{\beta a}}{\partial x^{c_1}}, \ldots, \frac{\partial^r \Gamma^{\alpha}_{\beta a}}{\partial x^{c_1} \ldots \partial x^{c_r}}; \Gamma^{i}_{jk}, \frac{\partial \Gamma^{i}_{jk}}{\partial x^{c_1}}, \ldots, \frac{\partial^s \Gamma^{i}_{jk}}{\partial x^{c_1} \ldots \partial x^{c_s}};\right.$$

$$\left. {}_{(E)}F^{\gamma h \ldots}_{\sigma l \ldots}, \frac{\partial_{(E)} F^{\ldots}_{\ldots}}{\partial x^{c_1}}, \ldots, \frac{\partial^t {}_{(E)} F^{\ldots}_{\ldots}}{\partial x^{c_1} \ldots \partial x^{c_t}}\right),$$

where ${}_{(E)}F^{\gamma h \ldots}_{\sigma l \ldots}$ are the components of any set of composite tensors, may be expressed as the same functions of arguments that are affine and composite tensors,

$$T^{\alpha a \cdots}_{\beta b \cdots} (0, A^{\alpha}_{\beta a c_1}, \ldots, A^{\alpha}_{\beta a c_1 \cdots c_r}; 0, A^{i}_{jkc_1}, \ldots, A^{i}_{jkc_1 \cdots c_s}; {}_{(E)}F^{\gamma h \cdots}_{\alpha l \cdots}, {}_{(E)}F^{\cdots}_{\cdots, c_1}, \ldots, {}_{(E)}F^{\cdots}_{\cdots, c_1 \cdots c_l}),$$

by replacing $\Gamma^{\alpha}_{\beta a}$ by zero, derivatives of $\Gamma^{\alpha}_{\beta a}$ by the corresponding König normal tensors, Γ^{i}_{jk} by zero, derivatives of Γ^{i}_{jk} by the corresponding affine normal tensors, and derivatives of the components of any composite tensor by the corresponding extension of that tensor.

This replacement theorem includes as a special case the replacement theorem for affine differential invariants.[4]

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THE STANDARD ERROR OF A TETRAD IN SAMPLES FROM A NORMAL POPULATION OF INDEPENDENT VARIABLES

BY S. S. WILKS*

DEPARTMENT OF MATHEMATICS, COLUMBIA UNIVERSITY

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A quantity which has been used extensively in testing the two-factor hypothesis of the mental abilities of man is the tetrad t_{1234} defined as $r_{12}r_{34} - r_{13}r_{24}$ where r_{ij} is the correlation between the i-th and j-th traits under observation. The mathematics involved in the theory of t_{1234} in small samples has offered a topic of research that has scarcely been touched. Thus far, several approximate expressions[1,2,3] have been obtained for the standard error of a tetrad. But they were derived by the older sampling theory as developed by Pearson, Filon and others and therefore lack the rigor which characterizes the modern theory of small samples. In 1928

Wishart[4] attacked the problem in the light of the small sample theory and considered the expression

$$P_{1234} = a_{12}a_{34} - a_{13}a_{24}$$

where a_{ij} is the covariance of the i-th and j-th traits. That is, $a_{ij} = r_{ij}s_is_j$ where r_{ij} is the sample value of the correlation coefficient between the i-th and j-th variates and s_i is the sample value of the standard deviation of the i-th variate. By a process of differentiation of the characteristic function of the variances and covariances, he found the exact expression for the expected value and standard error of P_{1234}.

In this note we shall find the exact expression for the standard error of t_{1234} in samples from a normal population in which the inter-correlations of the four variables are zero.

Let a sample ω of N items be drawn from a normal population of the four variables x_1, x_2, x_3 and x_4 whose distribution is

$$\frac{\sqrt{A_1A_2A_3A_4}}{\pi^2}\, e^{-\sum_{i=1}^{4} A_i(x_i - m_i)^2}, \tag{1}$$

where $A_i = \dfrac{1}{2\sigma_i^2}$, σ_i is the standard deviation and m_i the mean of x_i $(i = 1, 2, 3, 4)$. From the sample we form the following statistics

$$\bar{x} = \frac{1}{N}\sum_{\alpha=1}^{N} x_{i\alpha}$$

$$a_{ij} = \frac{1}{N}\sum_{\alpha=1}^{N}(x_{i\alpha} - \bar{x}_i)(x_{j\alpha} - \bar{x}_j), \quad (i, j = 1, 2, 3, 4),$$

where $x_{i\alpha}$ is the value of the i-th variate for the α-th individual of ω.

From Wishart's[5] generalized product moment distribution we have for the simultaneous distribution of the set $\{a_{ij}\}$ $(i, j = 1, 2, 3, 4)$:

$$f(\bar{a}) = \frac{[N^4A_1A_2A_3A_4]^{\frac{N-1}{2}}}{\pi^3\Gamma\left(\frac{N-1}{2}\right)\Gamma\left(\frac{N-2}{2}\right)\Gamma\left(\frac{N-3}{2}\right)\Gamma\left(\frac{N-4}{2}\right)}\, e^{-N\sum_{i=1}^{4} A_ia_{ii}}\, |a_{ij}|^{\frac{N-6}{2}} \tag{2}$$

where $|a_{ij}|$ is the determinant of the elements $\{a_{ij}\}$.

In order to find the standard error of t_{1234} we shall need the value of

$$\int f(\bar{a})e^{\alpha_{11}a_{11} + \alpha_{22}a_{22} + \alpha_{33}a_{33} + \alpha_{44}a_{44} + \beta_{12}a_{12} + \beta_{34}a_{34} + \beta_{13}a_{13} + \beta_{24}a_{24}}d\bar{a} \tag{3}$$

where $d\bar{a}$ is the product of the differentials of all of the a's in $\{a_{ij}\}$, and where the integration is taken over the field of all possible values of the a's.

The value of (3) follows at once from Wishart's[6] results, and it can be written as

$$\varphi = [N^4 A_1 A_2 A_3 A_4]^{\frac{N-1}{2}} \begin{vmatrix} NA_1 - \alpha_1 & -\frac{1}{2}\beta_1 & -\frac{1}{2}\beta_3 & 0 \\ -\frac{1}{2}\beta_1 & NA_2 - \alpha_2 & 0 & -\frac{1}{2}\beta_4 \\ -\frac{1}{2}\beta_3 & 0 & NA_3 - \alpha_4 & -\frac{1}{2}\beta_2 \\ 0 & -\frac{1}{2}\beta_4 & -\frac{1}{2}\beta_2 & NA_4 - \alpha_4 \end{vmatrix}^{-\frac{N-1}{2}}. \quad (4)$$

It can be shown from (4) that r_{12} and r_{34} are independently distributed. Indeed, if $\beta_3 = \beta_4 = 0$, (4) breaks into the product of two characteristic functions, each of which is the characteristic function of the variances and covariance in a sample from a bi-variate population. Therefore the joint distribution of r_{12} and r_{34} is the product of two functions, each of the form

$$\frac{\Gamma\left(\frac{N-1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{N-2}{2}\right)} (1 - t^2)^{\frac{N-4}{2}}. \quad (5)$$

The same properties hold for the distribution of the pair r_{13} and r_{24}. Since (5) is symmetrical with respect to $t = 0$, it is clear that the expected value of t_{1234} is zero. Hence, we have

$$\sigma^2_{t_{1234}} = E[t^2_{1234}] = E[r_{12}^2 r_{34}^2 - 2r_{12}r_{34}r_{13}r_{24} + r_{13}^2 r_{24}^2]. \quad (6)$$

Each of the terms $E[r_{12}^2 r_{34}^2]$ and $E[r_{13}^2 r_{24}^2]$ has the value

$$\frac{\Gamma^2\left(\frac{N-2}{2}\right)}{\Gamma^2\left(\frac{1}{2}\right)\Gamma^2\left(\frac{N-2}{2}\right)} \int_{-1}^{+1}\int_{-1}^{+1} \theta_1^2\theta_2^2(1 - \theta_1^2)^{\frac{N-4}{2}} (1 - \theta_2^2)^{\frac{N-4}{2}} d\theta_1 d\theta_2 = \frac{1}{(N-1)^2}. \quad (7)$$

The value of $E[-2r_{12}r_{34}r_{13}r_{24}]$ is, of course, the same as $-2E[r_{12}r_{34}r_{13}r_{24}]$. Now the expression defining $E[r_{12}r_{34}r_{13}r_{24}]$ can be obtained by integrating the integrand of (3) with respect to each of the α's from $-\infty$ to 0 and then differentiating the resulting form of the integrand once with respect to each of the four β's, setting the β's equal to zero afterward. It can be shown that, for values of the β's sufficiently near zero and for values of the α's such that $NA_i - \alpha_i > 0$ ($i = 1, 2, 3, 4$), the integrand of (3),

fulfills sufficient conditions[7] for any interchange of the order of the following operations: (*a*) integration with respect to the α's, (*b*) differentiation with respect to the β's and (*c*) integration with respect to the α's. Accordingly, we shall perform them in the order named. Thus, operation (*a*) yields (4). Applying operation (*b*) we find

$$\left.\frac{\partial^4\varphi}{\partial\beta_1\partial\beta_2\partial\beta_3\partial\beta_4}\right|_{\beta_1=\beta_2=\beta_3=\beta_4=0} = \frac{N-1}{16}(N^4A_1A_2A_3A_4)^{\frac{N-1}{2}}$$
$$[(NA_1-\alpha_1)(NA_2-\alpha_2)(NA_3-\alpha_3)(NA_4-\alpha_4)]^{-\frac{N+1}{2}}. \quad (8)$$

Integrating with respect to each of the α's from $-\infty$ to 0, we find the value of $E[r_{12}r_{34}r_{13}r_{24}]$ to be,

$$\frac{1}{(N-1)^3}. \quad (9)$$

Hence the value of $-2E[r_{12}r_{34}r_{13}r_{24}]$ is $-\frac{2}{(N-1)^3}$.

Therefore, we finally obtain as the exact formula for the standard error of t_{1234} in samples from the population (1),

$$\sigma_{t_{1234}} = \sqrt{\frac{2(N-2)}{(N-1)^3}}. \quad (10)$$

The use of the foregoing method for finding the exact value of the standard error of t_{1234} in samples from a general normal population of four variables leads to extremely complicated results which seem to defy simplification.

* NATIONAL RESEARCH FELLOW IN MATHEMATICS.

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